

# ON ENRICHING THE LEVIN-WEN MODEL WITH SYMMETRY

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**ABSTRACT.** Symmetry protected and symmetry enriched topological phases of matter are of great interest in condensed matter physics due to new materials such as topological insulators. The Levin-Wen model for spin/boson systems is an important rigorously solvable model for studying  $2D$  topological phases. The input data for the Levin-Wen model is a unitary fusion category, but the same model also works for unitary multi-fusion categories. In this paper, we provide the details for this extension of the Levin-Wen model, and show that the extended Levin-Wen model is a natural playground for the theoretical study of symmetry protected and symmetry enriched topological phases of matter.

## 1. INTRODUCTION

Symmetry protected and symmetry enriched topological phases of matter are of great interest in condensed matter physics due to new materials such as topological insulators (see [CGLW, BBCW] and references therein). The Levin-Wen (LW) model for spin/boson systems is an important rigorously solvable model for studying  $2D$  topological phases [LW]. The required input data for the LW model is a unitary fusion category (UFC), but the same model works for unitary multi-fusion categories. In this paper, we provide several results for this extension of the LW model, and show that the extended LW model is a natural playground for the theoretical study of symmetry protected and symmetry enriched topological phases of matter in two spatial dimensions.

The LW model is a Hamiltonian formulation of Turaev-Viro  $(2+1)$ -TQFTs. Three mathematical theorems underlie this beautiful model: (1) given a UFC  $\mathcal{C}$ , we can construct a Turaev-Viro unitary  $(2+1)$ -TQFT [BW], (2) the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  or quantum double  $D(\mathcal{C})$  of a UFC  $\mathcal{C}$  is always modular [Mü], and (3) the Turaev-Viro  $(2+1)$ -TQFT based on  $\mathcal{C}$  is equivalent to the Reshetikhin-Turaev

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$(2+1)$ -TQFT based on the center  $\mathcal{Z}(\mathcal{C})$  [BK, TV]. The algebraic model of anyons in the LW model with input  $\mathcal{C}$  is encoded by the modular category  $\mathcal{Z}(\mathcal{C})$ .

We conjecture that all three theorems above have appropriate extensions to unitary multi-fusion categories. Indeed the Drinfeld center  $\mathcal{Z}(\mathcal{C})$  of an indecomposable multi-fusion category  $\mathcal{C}$  is modular, and a direct sum of modular categories if  $\mathcal{C}$  is decomposable. Thus, we expect the Hilbert space  $V(S^2)$  of the 2-sphere  $S^2$  associated to a decomposable multi-fusion category  $\mathcal{C}$  has dimension  $> 1$ .

There are several generalizations of the LW model, including to  $3D$  and fermion systems [WW, GWW]. The first appearance of a LW model using a unitary multi-fusion category as input is given in Example  $H$  of Section III in [LWYW]. While the extension of the LW model to unitary multi-fusion categories as input is straightforward, the application of this extension to symmetry protected and symmetry enriched topological phases of matter is new.

In  $2D$ , the anyon model of a topological phase of quantum matter is algebraically modeled by a unitary modular category  $\mathcal{B}$ . An exciting new direction is the interplay between symmetry and topological order [BBCW]. But a microscopic physical theory based on local Hamiltonians is still lacking. For topological phases such that  $\mathcal{B}$  is a quantum double  $\mathcal{B} = D(\mathcal{C})$ , the LW model could provide such a microscopic theory. Specifically, given an input  $\mathcal{C}$  for the LW model, if the symmetry  $G$  could be realized as unitary on-site symmetries of the LW Hamiltonians, then the topological symmetry on  $D(\mathcal{C})$  should emerge from the  $G$  symmetry of the Hamiltonians. But even for the electric-magnetic duality  $e \leftrightarrow m$  of the toric code, a Hamiltonian realization is not in the literature<sup>1</sup>. Current realizations of the  $e \leftrightarrow m$  duality need the dual lattice and lattice translation.

In the case of a multi-fusion category, group symmetries sometimes appear in a natural way. For such a category it is natural to consider labels consisting of two indices. We may then endow the half-labels with a group structure  $G$ . Then the solutions of pentagons are closely related to  $G$ -equivariant 3-cocycles, and extended LW Hamiltonians sometimes naturally come with a  $G$ -symmetry, as we will see below. This leads to an application of the LW model to symmetry protected and symmetry enriched topological phases.

The contents of the paper are as follows: In Sec. 2, we provide some background material on multi-fusion categories. In Sec. 3, we give the detail of the extension of the LW model to multi-fusion category inputs and prove that the extended LW models with input  $\mathcal{M}_n$  all realize the trivial  $(2+1)$ -TQFT. In Sec. 4, we introduce group structures onto the half-label set of a multi-fusion category and use such group structures to enrich the LW model with symmetries. Finally, we de-equivariantize our  $G$ -symmetric LW models with a non-local transformation that leads to traditional LW models coupled with a local group action.

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<sup>1</sup>Meng Cheng found an on-site realization of the electric-magnetic duality in the toric code, but the details have not been published.

## 2. MULTI-FUSION CATEGORIES AND THEIR DOUBLES

All multi-fusion and modular categories in this paper are unitary over the complex numbers  $\mathbb{C}$ .

**2.1. Multi-fusion category.** The tensor unit is required to be a simple object in a fusion category. If we allow the tensor unit to be not necessarily simple, we obtain multi-fusion categories. Therefore, a multi-fusion category is a finite semi-simple rigid monoidal  $\mathbb{C}$ -linear category. They arise naturally in mathematics and physics. For example, given a finite depth type  $\Pi_1$  sub-factor  $N \subset M$  in the study of von Neumann algebras, the  $N - N$ ,  $N - M$ ,  $M - N$ , and  $M - M$  bi-modules form a Morita context, and can be regarded as a multi-fusion category. Much of the fusion category theory naturally generalizes to the multi-fusion case.

Given a multi-fusion category  $\mathcal{C}$  with a tensor unit  $\mathbf{1}$ , the tensor unit  $\mathbf{1}$  decomposes into the sum of simple objects  $\mathbf{1} \cong \bigoplus_{i=1}^n \mathbf{1}_i$  for some  $n$ . For a simple object  $X$  of  $\mathcal{C}$ , there exists a unique pair  $1 \leq i, j \leq n$  such that  $\mathbf{1}_i \otimes X \cong X \cong X \otimes \mathbf{1}_j$ . We will say that  $X$  is in the  $(i, j)$ -th component of  $\mathcal{C}$ . Let  $\mathcal{C}_{ij}$  be the abelian<sup>2</sup> sub-category of  $\mathcal{C}$  generated by direct sums of all simple objects in the  $(i, j)$ -th component. We will call  $\mathcal{C}_{ij}$  the  $(i, j)$ -th component of  $\mathcal{C}$ . The diagonal components  $\mathcal{C}_{ii}$  are fusion categories and the off-diagonal components  $\mathcal{C}_{ij}, i \neq j$ , are  $\mathcal{C}_{ii}$ - $\mathcal{C}_{jj}$ -bimodules. We will call such a multi-fusion category an  $n \times n$  multi-fusion category. A  $1 \times 1$  multi-fusion category is just a fusion category. A multi-fusion category is indecomposable if it is not the direct sum of two non-zero multi-fusion categories.

**Definition 2.1.** An  $n \times n$  **2-matrix** is an  $n \times n$  multi-fusion category for which each component  $\mathcal{C}_{i,j}$  is equivalent to  $\mathcal{V}ec$ , and the fusion rule is  $E_{ij} \otimes E_{kl} = \delta_{jk} E_{il}$ , where  $\{E_{ij}\}_{1 \leq i,j \leq n}$  is a complete set of isomorphism classes of all simple objects. We will call  $\{i\}_{1 \leq i \leq n}$  the half-label set.

**Example 2.2.** The  $n \times n$  **2-matrix**  $\mathcal{M}_n$ .

The multi-fusion category  $\mathcal{M}_n$  is the semi-simple category with simple objects  $\{E_{ij}\}, 1 \leq i, j \leq n$ , and fusion rule  $E_{ij} \otimes E_{kl} = \delta_{jk} E_{il}$ . The tensor product is strictly associative as matrix multiplication, and the tensor unit is  $\mathbf{1} = \bigoplus_{i=1}^n E_{ii}$ .  $\mathcal{M}_n$  can be regarded as a categorification of the matrix algebra  $M_n$  by replacing  $\mathbb{C}$  with  $\mathcal{V}ec$ .

A general object in  $\mathcal{M}_n$  is of the form  $X = \bigoplus_{i,j=1}^n x_{ij} E_{ij}, x_{ij} \in \mathbb{N}$ . The multiplicities  $x_{ij}$  will be assembled into an  $n \times n$  matrix, denoted also as  $X$ . So an object  $X$  is given by an  $n \times n$  matrix  $X = (x_{ij})_{1 \leq i,j \leq n}$  with non-negative integral entries, and  $E_{ij}$  is represented by the matrix as the notation indicates: all entries are zero except the  $(i, j)$ -entry, which is 1. Then the tensor product of two objects  $X, Y$  is

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<sup>2</sup>Here we mean “abelian” as in the sense it is used in category theory and homology theory, not as in abelian anyons.

just the matrix multiplication  $XY$ . For  $X = (x_{ij}), Y = (y_{ij})$ , a morphism from  $X$  to  $Y$  is of the form  $f = (f_{ij})$ , where  $f_{ij} : x_{ij}E_{ij} \rightarrow y_{ij}E_{ij}$  can be represented by a linear map from  $\mathbb{C}^{x_{ij}} \rightarrow \mathbb{C}^{y_{ij}}$ , or simply a  $y_{ij} \times x_{ij}$  matrix. Hence, a morphism in  $\mathcal{M}_n$  is simply a matrix of matrices. Then compositions of morphisms are given by entry-wise matrix multiplication.

**Example 2.3.** Morita contexts as multi-fusion categories.

Suppose  $\mathcal{C}$  is a fusion category and  $\mathcal{M}$  an indecomposable module category over  $\mathcal{C}$ . Let  $\mathcal{C}_{\mathcal{M}}^* = \text{Func}(\mathcal{M}, \mathcal{M})$  be the dual of  $\mathcal{C}$  with respect to  $\mathcal{M}$ . Then  $\begin{pmatrix} \mathcal{C} & \mathcal{M}^* \\ \mathcal{M} & \mathcal{C}_{\mathcal{M}}^* \end{pmatrix}$  is a  $2 \times 2$  multi-fusion category.

**2.2. Quantum Doubles.** Suppose  $\mathcal{C}$  is a multi-fusion category, then its quantum double  $D(\mathcal{C})$  in physics or Drinfeld center  $\mathcal{Z}(\mathcal{C})$  in mathematics is also a multi-fusion category. Note that  $D(\mathcal{C}_1 \oplus \mathcal{C}_2) \cong D(\mathcal{C}_1) \oplus D(\mathcal{C}_2)$  for two multi-fusion categories  $\mathcal{C}_i, i = 1, 2$ . Therefore, we will mainly focus on indecomposable multi-fusion categories.

**Theorem 2.4.** *Let  $\mathcal{C} = (\mathcal{C}_{ij})_{1 \leq i, j \leq n}$  be an  $n \times n$  indecomposable multi-fusion category. Then the quantum double  $D(\mathcal{C})$  of  $\mathcal{C}$  is equivalent to  $D(\mathcal{C}_{ii})$  for any  $1 \leq i \leq n$ . It follows that all  $\mathcal{C}_{ii}$  are categorically Morita equivalent to each other.*

*Proof.* If  $\mathcal{M}$  is an indecomposable module category over an indecomposable multi-fusion category  $\mathcal{C}$ , then  $D(\mathcal{C}) = D(\mathcal{C}_{\mathcal{M}}^*)$ , where  $\mathcal{C}_{\mathcal{M}}^*$  is the dual of  $\mathcal{C}$  with respect to  $\mathcal{M}$  (Corollary 3.35 [EO]). For a fixed  $i$ , let  $\mathcal{M}_i = \bigoplus_{k=1}^n \mathcal{C}_{ik}$ . Then  $\mathcal{M}_i$  is an indecomposable  $\mathcal{C}$ -module category. The dual category of  $\mathcal{C}$  with respect to  $\mathcal{M}_i$  is  $\mathcal{C}_{\mathcal{M}_i}^* \cong \mathcal{C}_{ii}^{\text{op}}$ , where  $\mathcal{C}_{ii}^{\text{op}}$  is the opposite category of  $\mathcal{C}$ . The theorem now follows from  $D(\mathcal{C}) \cong D(\mathcal{C}_{\mathcal{M}_i}^*) \cong D(\mathcal{C}_{ii}^{\text{op}}) \cong D(\mathcal{C})$ .  $\square$

**2.3. Doubles of  $n \times n$  2-matrices  $\mathcal{M}_n$ .** It follows from Thm. 2.4 that  $D(\mathcal{M}_n) \cong \text{Vec}$ . To keep our presentation elementary, we provide an explicit proof that  $D(\mathcal{M}_n)$  is  $\text{Vec}$  in this subsection.

Suppose  $X = (x_{ij}) = \bigoplus x_{ij}E_{ij}$  is an object of  $\mathcal{M}_n$ , and  $(X, c_X, -)$  an object of  $D(\mathcal{M}_n)$ . Then for any  $E_{ij}$ ,  $c_{X, E_{ij}} : X \otimes E_{ij} \rightarrow E_{ij} \otimes X$  is an isomorphism. Since  $X \otimes E_{ij} = \bigoplus_{k=1}^n x_{ki}E_{kj}$ , and  $E_{ij} \otimes X = \bigoplus_{k=1}^n x_{jk}E_{ik}$ , we have  $x_{ki} = 0, k \neq i$ , and  $x_{ii} = x_{jj}$  for any pair  $i, j$ . Write  $x_{ii} = m$ , then  $X \otimes E_{ij} = mE_{ij} = E_{ij} \otimes X$ , and  $c_{X, E_{ij}}$  is an  $n \times n$  matrix whose  $(i, j)$ -entry is an isomorphism  $mE_{ij} \rightarrow mE_{ij}$ , i.e. a matrix in  $GL(m, \mathbb{C})$ , and whose other entries are all 0. Thus an object of  $D(\mathcal{M}_n)$  is determined by the set  $\{(m, c_{ij})\}, 1 \leq i, j \leq n$ , where  $m$  is a positive integer, and  $c_{ij} \in GL(m, \mathbb{C})$ . Explicitly,  $X = mI_n$ , and the half braiding between  $X$  and  $E_{ij}$  is  $c_{ij} : mE_{ij} \rightarrow mE_{ij}$ .

To find the constraints from the hexagon equations as illustrated by Fig. 1, we see that the left-hand side of the equation in Fig. 1 is given by  $\delta_{jk}c_{il} : mE_{il} \rightarrow mE_{il}$ , and the right-hand side is given by  $\delta_{jk}c_{jl}c_{ij}$ . Thus we obtain

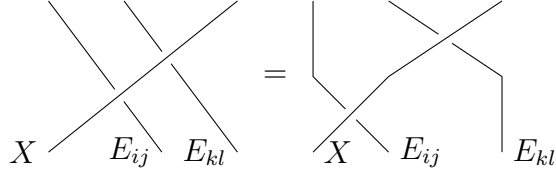


FIGURE 1. Hexagon Equations

$$(2.1) \quad c_{ij} = c_{kj}c_{ik}, \forall 1 \leq i, j, k \leq n.$$

Since every  $c_{ij}$  is invertible, it follows that  $c_{ii} = I_m$ , and  $c_{ij} = c_{ji}^{-1}$ . Hence the  $c_{ij}$ 's are completely determined by  $c_{i1}$ ,  $2 \leq i \leq n$  through the formula  $c_{ij} = c_{j1}^{-1}c_{i1}$ . The matrices  $c_{21}, \dots, c_{n1} \in GL(m, \mathbb{C})$  can be chosen arbitrarily, and  $c_{11} = I_m$ . Thus, an object of  $D(\mathcal{M}_n)$  is determined by a positive integer  $m$  and  $(n-1)$  matrices  $c_{21}, \dots, c_{n1} \in GL(m, \mathbb{C})$ .

To understand the morphisms in the doubles, we consider two objects  $(X, c_{ij}), (X', c'_{ij})$ , where  $X = mI_n, X' = m'I_n$ . Then a morphism  $\varphi : (X, c_{ij}) \rightarrow (X', c'_{ij})$  is given by  $(\delta_{ij}\varphi_{ii})$ , where  $\varphi_{ii} : mE_{ii} \rightarrow m'E_{ii}$  is a linear map. This morphism should commute with the half braiding, shown in Fig. 2.

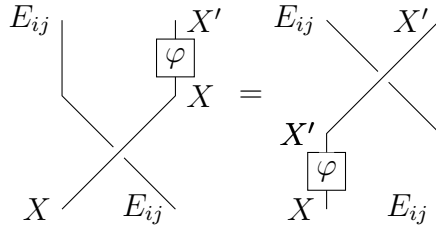


FIGURE 2. Morphisms in  $D(\mathcal{M}_n)$

Fig. 2 leads to the following equations for the morphism  $\varphi$  to satisfy:

$$\varphi_{jj}c_{ij} = c'_{ij}\varphi_{ii}.$$

Now assume  $m = m'$ , and  $\varphi_{ii}$  is an isomorphism. The equations above can be rewritten as  $c'_{ij} = \varphi_{jj}c_{ij}\varphi_{ii}^{-1}$ . By Eq. (2.1), it suffices to satisfy  $c'_{i1} = \varphi_{11}c_{i1}\varphi_{ii}^{-1}$  for  $i = 2, \dots, n$ . Using the freedom for choosing  $\varphi_{ii}$ , we choose them so that  $c'_{i1} = I_m$  for all  $i$ , and thus  $c'_{ij} = I_m, \forall 1 \leq i, j \leq n$ . Therefore, two objects of  $D(\mathcal{M}_n)$  are isomorphic if and only if their diagonal entries  $m$  and  $m'$  are the same, i.e. an isomorphism class is uniquely determined by a positive integer  $m$ . For each  $m$ , we choose a representative  $(X, c_{ij}) = (mI_n, I_m)$ , which is denoted as  $(m)$ .

Note that  $(m) \oplus (m') = (m + m')$ . Hence,  $D(\mathcal{M}_n)$  is generated by the single object  $(1) = (I_n, 1)$ . Note that  $\text{Hom}((1), (1)) = \mathbb{C}$ , so  $(1)$  is the only simple object in the category. Thus,  $D(\mathcal{M}_n) = \mathcal{Vec}$  as expected.

### 3. LEVIN-WEN MODEL FOR MULTI-FUSION CATEGORIES

Fix an integer  $d \geq 2$ , and a cellulation  $\gamma$  of an oriented closed surface  $Y$ . We often also refer to  $\gamma$  as a graph in  $Y$  by thinking about the 1-skeleton of  $\gamma$ . Let  $V(\gamma)$ ,  $E(\gamma)$ , and  $F(\gamma)$  be the set of vertices (sites), edges (bonds), and faces (plaquettes) of  $\gamma$ , respectively. Then  $L_\gamma(Y)$  will be the local Hilbert space  $\otimes_{e \in E(\gamma)} \mathbb{C}^d$ , i.e. we attach a *qudit*  $\mathbb{C}^d$  to each edge. The orthonormal basis of  $L_\gamma(Y)$  consists of all colors of the edges by a basis of  $\mathbb{C}^d$ . In this section,  $d$  will be the rank of the input UFC  $\mathcal{C}$ , i.e., the number of labels.

**Definition 3.1.** A Hamiltonian  $H$  is a *commuting local projector* (CLP) Hamiltonian if  $H = \sum_\alpha P_\alpha$ , where  $P_\alpha$  is a collection of pair-wise commuting local orthogonal projectors.

In general, we are not really interested in a single CLP Hamiltonian, rather a prescription for writing down a family of CLP Hamiltonians on all local Hilbert spaces  $L_\gamma(Y)$  associated to cellulations  $\gamma$  of  $Y$ . Such a prescription will be called a *Hamiltonian schema*. Since we are interested in thermodynamical physics, we need to study limits when the size of cellulations measured by the mesh goes to 0. We can use Pachner's theorem to organize all triangulations of a surface into a directed set. Then local Hilbert spaces and their ground state manifolds form inverse systems of finite dimensional Hilbert spaces.

The numerical data to specify the local Hilbert space and Hamiltonian of a LW model is a description of a UFC in terms of  $6j$ -symbols. In order to implement unitarity and symmetries, we demand some symmetries of the  $6j$  symbols. There are subtleties when the input UFC has multiplicities in the fusion rules, as defined below, and non-trivial Frobenius-Schur indicators. In the following, we will assume that all UFCs are multiplicity free and their modified  $6j$ -symbols, called tetrahedral symbols, have the full tetrahedral symmetry, as defined below. Not all UFCs have tetrahedral symbols that have the full tetrahedra symmetry [Ho].

**3.1. Levin-Wen Hamiltonian schema for unitary fusion categories.** A *label set*  $L$  is a finite set with a distinguished element 0 and with an involution  $^* : L \rightarrow L$  such that  $0^* = 0$ . Elements of  $L$  are called labels, 0 is called the trivial label, and  $j^* \in L$  is called the dual of  $j \in L$ .

A *fusion rule* on  $L$  is  $N : L \times L \times L \rightarrow \mathbb{N}$  such that for  $a, b, c, d \in L$ ,

$$(3.1) \quad N_{0a}^b = N_{a0}^b = \delta_{ab},$$

$$(3.2) \quad N_{ab}^0 = \delta_{ab^*},$$

$$(3.3) \quad \sum_{x \in L} N_{ab}^x N_{xc}^d = \sum_{x \in L} N_{ax}^d N_{cd}^x.$$

A fusion rule is *multiplicity-free* if  $N_{ab}^c \in \{0, 1\}$  for all  $a, b, c \in L$ . Set  $\delta_{abc} := N_{ab}^{c^*}$ , then  $\delta_{abc} = \delta_{bca}$  and  $\delta_{abc} = \delta_{c^*b^*a^*}$ . A triple  $(a, b, c)$  is admissible if  $\delta_{abc} = 1$ .

Given a fusion rule on  $L$ , a *loop weight* is a map  $w : L \rightarrow \mathbb{R} \setminus \{0\}$  such that  $w_{a^*} = w_a$  and

$$(3.4) \quad \sum_{c \in L} w_c \delta_{abc^*} = w_a w_b.$$

In particular,  $w_0 = 1$ . For unitary modular categories, the quantum dimensions—quantum traces of the identity morphisms—satisfy  $d_j \geq 1$  for all  $j \in L$ . Quantum dimensions might differ from loop weights  $\{w_i\}$ . We let  $\alpha_i = \frac{d_i}{w_i} = \pm 1$  for each label, and require:

$$(3.5) \quad \alpha_i \alpha_j \alpha_k = 1, \quad \text{if } \delta_{ijk} = 1.$$

A *symmetrized tetrahedral symbol* is a map  $T : L^6 \rightarrow \mathbb{C}$  satisfying the following conditions:

$$(3.6) \quad \text{tetrahedral symmetry:} \quad T_{kl n}^{ij m} = T_{nk^* l^*}^{mij} = T_{ijn^*}^{klm^*} = \alpha_m \alpha_n \overline{T_{l^* k^* n}^{j^* i^* m^*}},$$

$$(3.7) \quad \text{pentagon identity:} \quad \sum_n w_n T_{kp^* n}^{mlq} T_{mns^*}^{jip} T_{lkr^*}^{js^* n} = T_{q^* kr^*}^{jip} T_{m l s^*}^{riq^*},$$

$$(3.8) \quad \text{orthogonality condition:} \quad \sum_n w_n T_{kp^* n}^{mlq} T_{pk^* n}^{l^* m^* i^*} = \frac{\delta_{iq}}{w_i} \delta_{mlq} \delta_{k^* ip},$$

For convenience, we consider LW models defined on trivalent graphs in a closed oriented surface. Initially, we choose an arrow of each edge to assign a label, but the Hilbert space does not depend on these arrows, by using the following identification: for any state  $|\psi\rangle \in L_\gamma(Y)$ , if we reverse the direction of an edge  $e$  and replace its label  $j_e$  by its dual  $j_e^*$ , then the resulting state is identified with the initial state  $|\psi\rangle$ . See Fig. 3.

There are two types of local operators,  $Q_v$  which are defined at vertices  $v$  and  $B_p^s$  which are defined at a plaquette for an  $s \in L$ . Let us first define the operator  $Q_v$ . On a trivalent graph,  $Q_v$  acts on the labels of three edges incoming to the vertex  $v$ . We define the action of  $Q_v$  on the basis vector with  $j_1, j_2, j_3$  by

$$(3.9) \quad Q_v \left| \begin{array}{c} \nearrow j_3 \quad \nwarrow j_2 \\ \downarrow j_1 \end{array} \right\rangle = \delta_{j_1 j_2 j_3} \left| \begin{array}{c} \nearrow j_3 \quad \nwarrow j_2 \\ \downarrow j_1 \end{array} \right\rangle$$

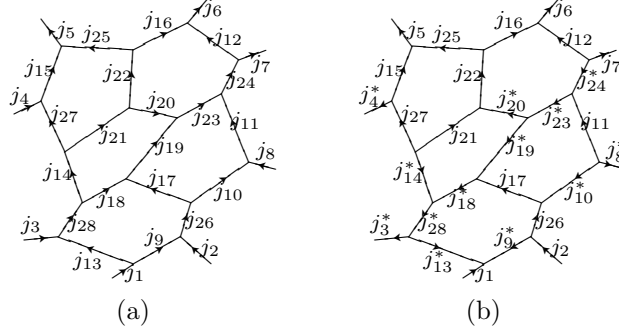


FIGURE 3. A configuration of string types on a directed trivalent graph. The configuration (b) is treated the same as (a), with some of the directions of some edges reversed and the corresponding labels  $j$  conjugated  $j^*$ .

where the tensor  $\delta_{j_1 j_2 j_3}$  equals either 1 or 0, which determines whether the triple  $(j_1, j_2, j_3)$  is “allowed” to meet at the vertex. Since  $\delta_{j_1 j_2 j_3} = \delta_{j_2 j_3 j_1}$ , the ordering in the three labels is not important. To be compatible with the conjugation structure of labels, the branching rule must satisfy  $\delta_{0 j j^*} = \delta_{0 j^* j} = 1$ ,  $\delta_{0 i j^*} = 0$  if  $i \neq j$ , and  $\delta_{j_1 j_2 j_3} = \delta_{j_3^* j_2^* j_1^*}$ .

One important property of the tetrahedral symbols is that

$$(3.10) \quad T_{kl n}^{i j m} = 0 \quad \text{unless } \delta_{i j m} = \delta_{k l m^*} = \delta_{l i n} = \delta_{n k^* j^*} = 1.$$

This is a consequence of the orthogonality condition and the tetrahedral symmetry.

For convenience, we take the square root of the loop weight as follows. We define

$$(3.11) \quad v_j := \frac{1}{T_{00j}^{j^* j 0}}.$$

We can verify  $v_j^2 = w_j$  from the orthogonality condition. In particular,  $v_0 = 1$ .

The operator  $B_p^s$  acts on the boundary edges of the plaquette  $p$ , and has the matrix elements on a triangle plaquette,

$$(3.12) \quad \left\langle \begin{array}{c} j_5 \quad j_3' \quad j_6 \\ \quad \swarrow \quad \downarrow \quad \searrow \\ j_1' \quad p \quad j_2' \\ \quad \swarrow \quad \downarrow \quad \searrow \\ j_4 \end{array} \right| B_p^s \left| \begin{array}{c} j_5 \quad j_3 \quad j_6 \\ \quad \swarrow \quad \downarrow \quad \searrow \\ j_1 \quad p \quad j_2 \\ \quad \swarrow \quad \downarrow \quad \searrow \\ j_4 \end{array} \right\rangle \\ = v_{j_1} v_{j_2} v_{j_3} v_{j_1'} v_{j_2'} v_{j_3'} T_{s j_3' j_1'}^{j_5 j_1^* j_3} T_{s j_1' j_2'}^{j_4 j_2^* j_1} T_{s j_2' j_3'}^{j_6 j_3^* j_2}.$$

The same rule applies when the plaquette  $p$  is a quadrangle, a pentagon, or a hexagon and so on. Note that the matrix is nondiagonal only on the labels of the boundary edges (i.e.,  $j_1$ ,  $j_2$ , and  $j_3$  on the above graph).



The operators  $B_p^s$  have the properties

$$(3.13) \quad B_p^{s\dagger} = B_p^{s*}$$

$$(3.14) \quad B_p^r B_p^s = \sum_t \delta_{rst*} B_p^t.$$

The Hamiltonian of the model is

$$(3.15) \quad H = - \sum_v Q_v - \sum_p B_p, \quad B_p = \frac{1}{D} \sum_s w_s B_p^s,$$

where  $D = \sum_j d_j^2$ , and the sum runs over all vertices  $v$  and all plaquettes  $p$  of the trivalent graph.

The main property of the interactions  $Q_v$  and  $B_p$  is that they are mutually-commuting, orthogonal projection: (1)  $[Q_v, Q_{v'}] = 0 = [B_p, B_{p'}], [Q_v, B_p] = 0$ ; (2)  $Q_v^2 = Q_v = Q_v^*$  and  $B_p^2 = B_p = B_p^*$ . Thus the Hamiltonian is exactly soluble. The elementary energy eigenstates are given by common eigenvectors of all these projections. The ground states have eigenvalues  $Q_v = B_p = 1$  for all  $v$  and  $p$ , while each excited state violates these constraints for some subset of the plaquettes and vertices.

**3.2. Multi-fusion category extension of the Levin-Wen model.** The input data for LW models can be extended to the multi-fusion case. The extension is to replace the trivial label 0 by a subset  $L_0$  of  $L$ , in order to numerically specify the (not necessarily simple) tensor unit of the category.

We start with a *label set*  $L$  with an involution  $*$  :  $L \rightarrow L$  that is equipped with a trivial set  $L_0$ , where  $L_0$  is determined by the decomposition of the tensor unit into simple objects as in Sec. 2.1. A *fusion rule* on  $L$  is a map  $N : L \times L \times L \rightarrow \mathbb{N}$  satisfying that for all  $a, b, c, d \in L$ ,

$$(3.16) \quad \sum_{\alpha \in L_0} N_{\alpha a}^b = \sum_{\alpha \in L_0} N_{a\alpha}^b = \delta_{ab},$$

$$(3.17) \quad \sum_{\alpha \in L_0} N_{ab}^\alpha = \delta_{ab*},$$

$$(3.18) \quad \sum_{x \in L} N_{ab}^x N_{xc}^d = \sum_{x \in L} N_{ax}^d N_{cd}^x.$$

These three equations are obtained by formally replacing 0 by  $\sum_{\alpha \in L_0} \alpha$  in Eqs. (3.1), (3.2) and (3.3). Since  $N_{a\alpha}^b \in \mathbb{N}$ , the first equality implies that for each label  $a \in L$ , there exists a unique pair  $(\alpha, \beta) \in L_0 \times L_0$  such that  $N_{\alpha'a}^b = \delta_{ab} \delta_{\alpha'\alpha}$  and  $N_{a\beta'}^b = \delta_{ab} \delta_{\beta'\beta}$  for  $b \in L, \alpha', \beta' \in L_0$ . We say  $a$  has the grading  $(\alpha, \beta)$ . Obviously, each  $\alpha \in L_0$  has the grading  $(\alpha, \alpha)$ .

Therefore  $L$  is graded by  $L_0 \times L_0$ :  $L = \bigsqcup_{\alpha, \beta \in L_0} \alpha L_\beta$ , and we can denote the labels in  ${}_\alpha L_\beta$  by  ${}_\alpha a_\beta$  to specify their gradings  $(\alpha, \beta)$ . Eqs. (3.16) and (3.18) imply

$$(3.19) \quad N_{\alpha a_\beta, \gamma b_\delta}^{\epsilon c_\zeta} = 0 \quad \text{unless } \alpha = \epsilon, \beta = \gamma, \delta = \zeta.$$

Together with Eq. (3.17), it implies

$$(3.20) \quad \alpha^* = \alpha \quad \text{for } \alpha \in L_0,$$

$$(3.21) \quad {}_\alpha a_\beta^* \in {}_\beta L_\alpha \quad \text{for } {}_\alpha a_\beta \in {}_\alpha L_\beta.$$

Given a fusion rule on  $\{L, L_0\}$ , the loop weight satisfies

$$(3.22) \quad \sum_{\alpha c_\gamma \in {}_\alpha L_\gamma} \delta_{\alpha a_\beta, \beta b_\gamma, (\alpha c_\gamma)^*} w_{\alpha c_\gamma} = w_{\alpha a_\beta} w_{\beta b_\gamma}.$$

The symmetrized tetrahedral symbols are defined in the same way as those in the previous section, and so are the LW models. This leads to the following conclusion:

**Proposition 3.2.** *Using the modified label set  $L$  with trivial set  $L_0$ , the LW Hamiltonian schemas extend to multi-fusion categories, and all resulting Hamiltonians are CLPs.*

**3.3. The  $n \times n$  2-matrix  $\mathcal{M}_n$  as input.** Consider the multi-fusion category  $\mathcal{M}_n$  from example 2.2. This example gives the following data. The label set is  $L = \{E_{ij}\}$ , the trivial set is  $L_0 = \{E_{ii}\}$ , and the fusion rule is

$$(3.23) \quad \delta_{E_{ij}, E_{kl}, E_{mn}} = \delta_{jk} \delta_{lm} \delta_{ni}.$$

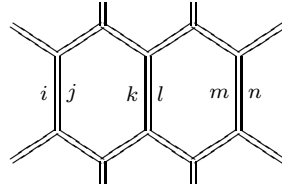
The set  $L = \bigsqcup_{i,j} {}_i L_j$  is graded by  $i, j$  where each  ${}_i L_j$  has only one element,  $E_{ij}$ . The duals are  $E_{ij}^* = E_{ji}$ .

Let us set the loop weights to be  $w_{E_{ij}} = 1$  for all  $i, j$ . The simplest normalized 6j-symbol is to take

$$(3.24) \quad T_{def}^{abc} = \begin{cases} 1 & \text{if } \delta_{abc} = \delta_{dec^*} = \delta_{eaf} = \delta_{fd^*b^*} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

for  $a, b, c, d, e, f \in L$ .

The local Hilbert space is spanned by labels on all edges. In our example, labels are the gradings  $(i, j)$ . Graphically, we use a double line to represent the gradings as illustrated below.



We do not draw arrows in the graph as a label on each arrowed edge is identified with its dual on the same edge with the arrow reversed. For example, the labels

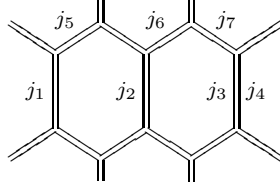
on the three vertical edges illustrated above read as  $E_{ij}$ ,  $E_{kl}$  and  $E_{mn}$  upwards, and as  $E_{ji}$ ,  $E_{lk}$  and  $E_{nm}$  downwards.

Consider the eigenspace  $\mathcal{L}^{Q=1}$  of  $Q_v = 1$  for all vertices. The fusion rule in Eq. (3.23) has a double line representation near each vertex of the form



which presents an admissible triple  $(E_{ij}, E_{jk}, E_{kl})$  on the three edges incoming into the vertex, and for which all other combinations are not allowed. If two lines are connected, then they carry the same label  $i$ .

Therefore the basis vectors in  $\mathcal{L}^{Q=1} = \otimes_p \mathbb{C}^n$  have a double line representation as below.



To each plaquette  $p$ , there is a loop labeled by  $j_p$ . The basis is denoted in terms of the loop labels  $j_p$  and given by  $\{|j_1, j_2, \dots\rangle\}$ . This statement holds for the model on any closed surface.

The operator  $B_p$  is now  $B_p = \frac{1}{n} \sum_{\alpha\beta} B_p^{E_{\alpha\beta}}$ , where  $B_p^{E_{\alpha\beta}}$  is defined in Eq. (3.12). In the subspace  $\mathcal{L}^{Q=1}$ ,  $B_p^{E_{\alpha\beta}}$  is a map

$$(3.25) \quad B_p^{E_{\alpha\beta}} : |j_1, j_2, \dots, j_p, \dots\rangle \mapsto \delta_{\beta, j_p} |j_1, j_2, \dots, \alpha, \dots\rangle.$$

Therefore there is only one ground state, with common eigenvalues  $Q_v = 1$  and  $B_p = 1$  for all  $v, p$ :

$$(3.26) \quad |\Phi\rangle = \sum_{\alpha_1, \alpha_2, \dots} |\alpha_1, \alpha_2, \dots, \alpha_p, \dots\rangle,$$

up to a constant normalization factor. The discussion can be summarized by the following proposition.

**Proposition 3.3.** *The LW Hamiltonian schemas with input  $\mathcal{M}_n$  for all  $n \geq 1$  realize the trivial  $(2+1)$ -TQFT.*

Consider now the example  $n = 2$ , for which it is easy to give an explicit description of the ground state. In this case the operator  $B_p$  is the matrix  $\frac{1}{2}(\mathbf{1} + \sigma^x)$  in the local basis  $|i_p\rangle$ , where  $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is a Pauli matrix. Dropping the constant terms, we can write the Hamiltonian in the subspace  $\mathcal{L}^{Q=1}$  as

$$(3.27) \quad H|_{Q=1} = -\frac{1}{2} \sum_p \sigma_p^x.$$

It is convenient to use the dual graph picture. Namely, by taking the dual graph of a spatial trivalent graph, we obtain a triangulation of the surface. Then the ground state is simply a tensor product  $\otimes_p |\sigma_p^x = 1\rangle$  of all local eigenstates of  $\sigma^x = 1$  at the vertices of the dual triangulation.

**3.4. Degeneracy on a Disk.** Consider the disk with a smooth loop boundary. On the graph in Fig. 4(a), the Hamiltonian takes the form in Eq. (3.15), with the first summation over all vertices of the graph and over all internal plaquettes inside the disk.

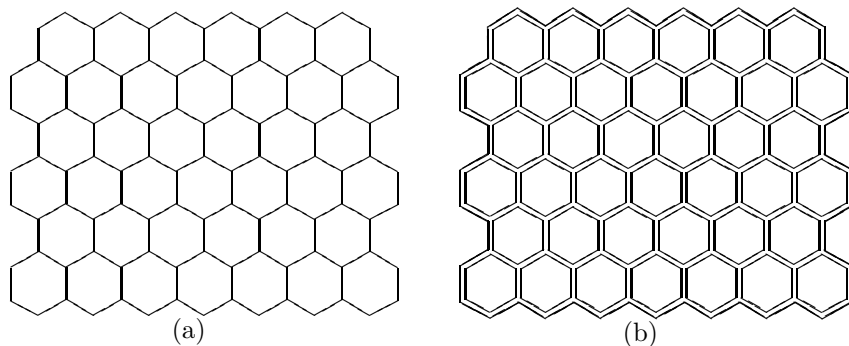


FIGURE 4. (a). Disk with a loop boundary. (b). Double line representation for  $\mathcal{L}^{Q=1}$ .

The double line representation for  $\mathcal{L}^{Q=1}$  is illustrated in Fig. 4(b). A basis vector in  $\mathcal{L}^{Q=1}$  is denoted by  $|\alpha_\partial; \alpha_1, \alpha_2, \dots, \alpha_p, \dots\rangle$ , specified by a loop value  $\alpha_p$  associated to each plaquette  $p$  inside the disk, and a loop value  $\alpha_\partial$  associated to the boundary.

The second term  $-\sum_p B_p$  in the Hamiltonian does not affect  $\alpha_\partial$ . Therefore, the ground states are degenerate and parameterized by  $\alpha_\partial$ . For the input data  $\mathcal{M}_n$ , the ground state degeneracy is  $n$ .

Similar to the formula in Eq.(3.26), the degenerate ground states for all  $\alpha_\partial$  are

$$(3.28) \quad |\Phi(\alpha_\partial)\rangle = \sum_{\alpha_1, \alpha_2, \dots} |\alpha_\partial; \alpha_1, \alpha_2, \dots, \alpha_p, \dots\rangle$$

**3.5. Topological Entanglement Entropy.** Consider the extended LW model with  $\mathcal{M}_n$  as input. We divide a trivalent graph into two subsystems  $A$  and  $B$ , where their boundary intersects some edges, denoted by a dashed curve as illustrated in Fig. 5.

Denote the edges across the boundary by  $j_1, j_2, \dots, j_l \in L$ , or simply  $\{j_i\}$  for short. The number  $l$  will be called the length of the boundary curve.



By symmetry,  $\rho_A$  has  $n^l$  equal eigenvalues, which are normalized to  $\lambda = 1/n^l$  by the trace condition  $\text{tr}_A(\rho_A) = 1$ . It follows that

$$(3.31) \quad S_E = \log(n)l.$$

Since there is not any sub-leading correction term in  $S_E$  — it is exactly proportional to the length  $l$  of the boundary curve — the topological entanglement entropy is 0 [KP, LW2]. A similar calculation on the torus also leads to zero topological entanglement entropy.

#### 4. SYMMETRY ENRICHING THE LEVIN-WEN MODEL

We are interested in enriching the LW model with on-site unitary symmetries. A good example is the toric code Hamiltonian  $H = -\sum_v A_v - \sum_p B_p$  on the square lattice, where a qubit is one each edge. As usual, the vertex operator  $A_v$  is the tensor product of  $\sigma^x$  and the identity, while the plaquette term is a tensor product of  $\sigma^z$  and the identity. A moment's thought shows that the tensor product of  $\sigma^x$  (or  $\sigma^z$ ) over all edges is an on-site unitary symmetry of the toric code Hamiltonian. Of course this  $\mathbb{Z}_2$  symmetry is very trivial because it will not permute anyon types. But even if a  $\mathbb{Z}_2$  symmetry of the toric code does not permute anyon types, there are still four different ways to fractionalize a  $\mathbb{Z}_2$  symmetry in a one-to-one correspondence to classes in  $H^2(\mathbb{Z}_2; \mathbb{Z}_2^2) = \mathbb{Z}_2^2$  [BBCW]. In this section, we will describe analogous symmetries of the LW Hamiltonians. It will be interesting to understand their role in a microscopic theory of symmetry fractionalization, symmetry defects, and gauging using fixed-point rigorously solvable Hamiltonians.

**4.1. Classification of  $n \times n$  2-matrices.** The half-label set can be endowed with a group structure. In this subsection, we classify all  $n \times n$  2-matrices whose half-label set has the structure of an abelian group  $G$ .

By the fusion rule, there are four independent variables in the 6j-symbols. Denote them by

$$(4.1) \quad \phi_4(\alpha, \beta, \gamma, \delta) := T_{E_{\gamma\delta}E_{\delta\alpha}E_{\beta\delta}}^{E_{\alpha\beta}E_{\beta\gamma}E_{\gamma\delta}} w_{E_{\beta\delta}}.$$

In this notation the pentagon identity can be written as

$$(4.2) \quad \phi_4(\alpha, \beta, \gamma, \delta) \phi_4(\alpha, \beta, \delta, \epsilon) \phi_4(\beta, \gamma, \delta, \epsilon) = \phi_4(\alpha, \gamma, \delta, \epsilon) \phi_4(\alpha, \beta, \gamma, \epsilon),$$

for  $\alpha, \beta, \gamma, \delta = 1, 2, \dots, n$ .

Suppose the half labels  $\alpha, \beta, \dots$  form a finite group  $G$  with  $|G| = n$ , e.g.  $G = \mathbb{Z}_n$ . Recall that a homogeneous  $n$ -cochain taking values in  $\mathbb{C}$  is a map  $\phi_{n+1} : G^{n+1} \rightarrow \mathbb{C} \setminus \{0\}$  such that  $g \cdot \phi_{n+1}(g_1, \dots, g_{n+1}) = \phi_{n+1}(gg_1, \dots, gg_{n+1})$ . We will usually consider the trivial  $G$ -action on  $\mathbb{C} \setminus \{0\}$ . Hence,  $\phi_4 : G^4 \rightarrow \mathbb{C} \setminus \{0\}$  is a homogeneous 3-cochain on  $G$ , equipped with an action:

$$(4.3) \quad g \cdot \phi_4(\alpha, \beta, \gamma, \delta) = \phi_4(g\alpha, g\beta, g\gamma, g\delta),$$

where we regard  $\mathbb{C} \setminus \{0\}$  as a trivial  $G$ -module. The pentagon identity (4.2) can then identified with the 3-cocycle condition  $\delta\phi_4 = 1$ , where the coboundary  $\delta$  is defined by

$$(4.4) \quad \delta\phi_4(\alpha_0, \alpha_1, \dots, \alpha_4) = \prod_{0 \leq i \leq 4} \phi_4(\alpha_0, \alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_4)^{(-1)^i}.$$

Therefore, the  $6j$ -symbols are classified by the third group cohomology classes in  $H^3(G, U(1))$ . Note that not all 3-cocycles satisfy the tetrahedral symmetry in Eq. (3.6). We call 3-cocycles  $\phi_4$  defined as above  $G$ -invariant.

**Definition 4.1.** Given a finite group  $G$  and a homogeneous 3-cocycle  $\phi_4$ ,  $\phi_4$  is called  $G$ -invariant if  $\phi_4(\alpha, \beta, \gamma, \delta) = \phi_4(g\alpha, g\beta, g\gamma, g\delta)$  for all  $\alpha, \beta, \gamma, \delta = 1, \dots, n$ , and  $g \in G$ . I.e. the action of  $G$  on  $\phi_4$  given by Eq. (4.3) is trivial if  $\mathbb{C} \setminus \{0\}$  is regarded as a trivial  $G$ -module.

Consider the case where  $n = 2$ . Then the group is  $\mathbb{Z}_2 = \{0, 1\}$ . There are two equivalence classes, with the 3-cocycle representatives:

- (1)  $w_{E_{\alpha\beta}} = 1$ , and  $\phi_4 = 1$  is constant, as in Sec. 3.3;
- (2)  $w_{E_{\alpha\beta}} = \begin{cases} 1 & \text{if } \alpha = \beta \\ -1 & \text{if } \alpha \neq \beta \end{cases}$ , and

$$\phi_4(\alpha, \beta, \gamma, \delta) = \exp \left[ \frac{\pi i}{2} (2 - |\alpha + \beta + \gamma + \delta - 2|) \right] w_{E_{\beta\delta}}.$$

The two representatives are chosen to satisfy the tetrahedral symmetry in Eq. (3.6). The  $G$ -actions in Eq. (4.3) on both 3-cocycles are trivial, hence both 3-cocycles are  $\mathbb{Z}_2$ -invariant.

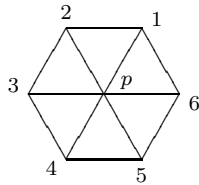
Similar to Eq. (3.27), the Hamiltonian for the second class can be written as

$$(4.5) \quad H = -\frac{1}{2} \sum_p \tau_p^x.$$

In the dual triangulation,  $\tau^x$  is

$$(4.6) \quad \tau^x = \left\{ \prod_{\langle ij \rangle \in \partial p} \exp \left[ i \frac{\pi}{4} (\mathbf{1} - \sigma_i^z \sigma_j^z) + i \frac{\pi}{2} (\mathbf{1} + \sigma_i^z \sigma_j^z) \right] \right\} \sigma_p^x,$$

with the product over nearest neighbor vertex pairs on the boundary of  $p$ , for example, over  $\langle 12 \rangle, \langle 23 \rangle, \dots, \langle 61 \rangle$  in the example below:



Here only the relevant triangles of the dual graph are shown, assuming the remaining part of the graph is not affected.

**4.2.  $G$ -symmetric Hamiltonian Schema.** Given a homogeneous 3-cocycle  $\phi_4$ , not necessarily  $G$ -invariant, we have a multi-fusion category  $(\mathcal{M}_n, \phi_4)$  with  $6j$ -symbols given by Eq. (4.1). This in turn allows us to define a Levin-Wen Hamiltonian schema with this multi-fusion category as input.

**Definition 4.2.** Given a finite group  $G$  and a Levin-Wen Hamiltonian schema, the Levin-Wen Hamiltonian schema is  $G$ -symmetric if each  $g \in G$  acts on the qudit  $\mathbb{C}^d$  as a unitary matrix  $U_g$ , such that it is a symmetry of all resulting Levin-Wen Hamiltonians.

**Theorem 4.3.** *If the homogeneous 3-cocycle  $\phi_4$  for an  $n \times n$   $\mathbf{2}$ -matrix is  $G$ -invariant, then the Levin-Wen Hamiltonian schema with the  $n \times n$   $\mathbf{2}$ -matrix  $(\mathcal{M}_n, \phi_4)$  input is  $G$ -symmetric, and realizes a  $G$ -symmetry protected topological phase (SPT).*

Using Prop. 3.3, we just need to check the  $G$ -invariance of Levin-Wen Hamiltonians, which is a straightforward check. But it is not clear if we have realized any non-trivial SPTs, which will be addressed in the next section.

We conjecture that this result can be extended in the following way.

**Conjecture 4.4.** *The LW Hamiltonian schema with an  $n \times n$  multi-fusion  $\mathcal{C}$  input realizes a symmetry enriched topological phase  $D(\mathcal{C})$  with some on-site unitary symmetry  $G$ , which does not permute anyon types.*

**4.3. De-equivariantizing the  $G$ -symmetric Levin-Wen model.** To understand if the SPTs realized in Thm.4.3 are non-trivial, we study the gauging of the symmetry  $G$  [LG, BBCW]. First we give a proof of the following proposition.

**Proposition 4.5.** *There is a non-local transformation from  $G$ -symmetric LW models to traditional LW models coupled to a local action.*

Given a finite group  $G$ , a homogeneous 3-cocycle  $\phi_4$  of  $G$  can be de-equivariantized to obtain an inhomogeneous 3-cocycle  $\varphi_3$  by setting

$$(4.7) \quad \varphi_3(x, y, z) = \phi_4(1, x, xy, xyz),$$

for  $x, y, z \in G$  and 1 is the identity element of  $G$ . The 3-cocycle  $\varphi_3$  has a group action

$$(4.8) \quad g \cdot \varphi_3(x, y, z) = \phi_4(g, gx, gxy, gxyz).$$

The inhomogeneous 3 cocycles  $\varphi_3$  and homogeneous 3-cocycles  $\phi_4$  are in one-one correspondence because  $\phi_4$  can be recovered from  $\varphi_3$  by

$$(4.9) \quad \phi_4(\alpha, \beta, \gamma, \delta) = \alpha \cdot \varphi_3(\alpha^{-1}\beta, \beta^{-1}\gamma, \gamma^{-1}\delta).$$



This de-equivariantization reduces the  $G$ -symmetric data from a multi-fusion category to input data from an abelian modular category  $\mathcal{Vec}_G^{\varphi_3}$  with a nontrivial action of  $G$  on  $\varphi_3$ .

The correspondence between  $\phi_4$  and  $\varphi_3$  can be adapted to the local Hilbert spaces and their Hamiltonians, therefore, the correspondence establishes a non-local duality transformation. In the following, we will work with the dual triangulations and consider only the 2-sphere  $S^2$  for simplicity.

For the local Hilbert spaces, the subspaces  $\mathcal{L}^{Q=1}$  are spanned by the group elements  $\{\alpha_p\}$  at vertices  $p$  of the dual triangulations. Choose an arbitrary vertex  $p_0$ , and designate it as the *origin*.

On the 2-sphere, the set of group elements  $\{\alpha_0, \alpha_1, \alpha_2, \dots\}$  assigned to vertices corresponds to the set of group elements  $\{g_1, g_2, \dots\}$  assigned to edges satisfying the following condition: around any triangle, the holonomy (the product of the three group elements around the triangle) is equal to the identity 1. In fact, the group element  $g_e$  on each edge  $e$  can be written as  $g_e = \alpha_2 \alpha_1^{-1}$ , so it is determined by  $\alpha_1$  ( $\alpha_2$ ) at the starting (ending) point of  $e$ . Conversely, given  $\alpha_0$  at the origin vertex  $p_0$ ,  $\alpha_p$  can be determined as follows: choose an arbitrary path from  $p_0$  to  $p$ , multiply the group elements on the edges along the path and  $\alpha_0$ . The two constructions above give rise to an isomorphism

$$(4.10) \quad \{\alpha_0, \alpha_1, \alpha_2, \dots\}|_{\text{vertex colors}} \cong \{\alpha_0; g_1, g_2, \dots\}|_{\text{trivial holonomy}}.$$

where “trivial holonomy” means that the group elements  $g$  around each triangle have a product equal to the identity 1. Therefore, the Hilbert space  $\mathcal{L}^{Q=1}$  has a basis

$$(4.11) \quad \{|\alpha_0; g_1, g_2, \dots\rangle\}|_{\text{trivial holonomy}}$$

If the  $G$ -action is trivial, then the  $G$ -symmetric Hamiltonian can be de-equivariantized as follows. First,  $\varphi_3$  produces new input data  $\{\tilde{w}, \tilde{\delta}, \tilde{T}\}$ , where  $g, g_1, g_2, g_3 \in G$ , by defining

$$(4.12) \quad \tilde{w}_g = w_{E_{1g}},$$

$$(4.13) \quad \tilde{\delta}_{g_1, g_2, g_3} = \delta_{g_1 g_2 g_3, 1},$$

$$(4.14) \quad \tilde{T}_{g_3, (g_1 g_2 g_3)^{-1}, g_2 g_3}^{g_1, g_2, (g_1 g_2)^{-1}} = \varphi_3(g_1, g_2, g_3) / w_{g_2 g_3}.$$

Then, the Hamiltonian in terms of  $\{\tilde{w}, \tilde{\delta}, \tilde{T}\}$  is

$$(4.15) \quad H = - \sum_v \tilde{Q}_v - \sum_p \tilde{B}_p,$$

where  $\tilde{B}_p = \frac{1}{n} \sum_g w_g \tilde{B}_p^g$  for all plaquettes except for  $p_0$ , and  $\tilde{B}_p^g$  is defined as in Eq. (3.12) in terms of  $\varphi_3$ , which acts on the degrees of freedom  $g_1, g_2, \dots$  in the basis (4.11).

At  $p_0$ ,  $\tilde{B}_{p_0} = \frac{1}{n} \sum_g w_g \tilde{B}_{p_0}^g T_{p_0}^g$ , where

$$(4.16) \quad T_{p_0}^g : |\alpha_0; g_1, g_2, \dots\rangle \mapsto |g\alpha_0; g_1, g_2, \dots\rangle.$$

Therefore, the non-local transformation defines a one-to-one correspondence between the  $G$ -symmetric LW models and the modified traditional LW models with input data from  $\mathcal{Vec}_G^{\varphi_3}$  and  $\tilde{B}_{p_0}$  coupled to the local group action  $T_{p_0}^g$ . The local group action  $T_{p_0}^g$  corresponds to a global action in the  $G$ -symmetric LW model:

$$(4.17) \quad T^g : |\alpha_0, \alpha_1, \dots, \alpha_p, \dots\rangle \mapsto |g\alpha_0, g\alpha_1, \dots, g\alpha_p, \dots\rangle$$

Let us apply the non-local transformation on the ground state  $\Phi$  on the 2-sphere. In the transformed traditional LW model, the ground state is the common eigenstate of  $\tilde{B}_p^g = 1$ , for  $p \neq p_0$ , and  $\tilde{B}_{p_0}^g T_{p_0}^g = 1$ , for all  $g \in G$ . The global constraint in the traditional LW model enforces  $\tilde{B}_{p_0}^g = 1$  and hence  $T_{p_0}^g = 1$ . By the non-local transformation,  $T_{p_0}^g = 1$  means that the ground state is invariant under the global symmetry  $\{T^g\}$  in the  $G$ -symmetric LW model.

**Physical Theorem**<sup>3</sup>: *The  $G$ -symmetric LW model with input  $\mathcal{M}_n$  realizes a  $G$ -SPT with the 3-cocycle  $\varphi_3 \in H^3(G; U(1))$  when  $\varphi_3$  is  $G$ -invariant.*

We did not prove this theorem mathematically because we did not define universality classes of SPT phases mathematically. But physically we summarize the argument above as follows. Each  $G$ -invariant 3-cocycle  $\varphi_3$  leads to an SPT because the LW model realizes the trivial TQFT. To understand the local term  $T_{p_0}^g$ , we map the SPT model to a nontrivial TQFT coupled to a gauge field with a gauge coupling term, where the half-labels represent the gauge field. If we eliminate the gauge coupling term, all half-labels are eliminated as well except the one at the base point. This leaves behind the local term at the base point.

**Remark 4.6.** The input  $6j$ -symbols in Eqs. (4.12)-(4.14) are well-defined only when the  $G$ -action on  $\varphi_3$  is trivial. So de-equivariantization works only for trivial  $G$ -actions. If the  $G$ -action on  $\varphi_3$  is nontrivial, then the  $6j$ -symbols are equipped with a  $G$ -action, which leads to a LW model with a gauge group action.

**4.4. On a Disk.** Consider further a disk with a smooth boundary, e.g., with the graph in Fig. 4(a). The non-local transformation leads to the same form of the Hamiltonian as in Eq. (4.15), but with the second summation over all plaquettes  $p$  inside the disk. The degenerate ground states  $\Phi(\alpha_\partial)$  in the  $G$ -symmetric LW model are parameterized by the half-label  $\alpha_\partial$ . Now let us reexamine the ground states in the traditional LW model under the non-local transformation.

Take an arbitrary plaquette inside the disk as the origin, denoted by  $p_0$ . The ground states are the common eigenstates of  $\tilde{B}_p^g = 1$ , for  $p \neq p_0$  inside the disk, and  $\tilde{B}_{p_0}^g T_{p_0}^g = 1$ , for all  $g \in G$ . Due to the presence of the boundary, the global

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<sup>3</sup>By a physical theorem, we mean that the argument is only rigorous physically. Therefore, physical theorems should be regarded as mathematical conjectures.

constraint on  $\tilde{B}_{p_0}^g$  is released. If  $\tilde{B}_{p_0}^g$  transforms under a non-trivial irreducible representation  $\rho$  of  $G$ , we say there is an elementary quasiparticle (or a topological defect) at  $p_0$  identified by its topological charge  $\rho$ . This topological charge is always coupled to a charge which transforms under the dual representation  $\rho^*$  of the local group action.

The degenerate ground states  $\Phi_\rho$  are thus parametrized by the charge  $\rho$ . Under the non-local transformation, they correspond to the ground states in the  $G$ -symmetric LW model, carrying a global charge  $\rho^*$  under the global symmetry  $\{T^g\}$ . Meanwhile, the topological charge  $\rho$  of the local quasiparticle in the traditional LW model is mapped to the boundary condition specified by  $\rho$  in the  $G$ -symmetric LW model. This relation between  $G$ -symmetric LW models and LW models coupled to a gauge action is listed in Table 1.

TABLE 1. Non-local transformation on a disk.

$G$ -symmetric LW model	Traditional LW model coupled to a local action
global symmetry	a local action on Hamiltonians
boundary condition specified by $\rho$	bulk local quasiparticle with topological charge $\rho$
global charge $\rho^*$	a local charge $\rho^*$ coupled to the quasiparticle

For example, take  $\mathcal{M}_n$  as the input data, and let  $G = \mathbb{Z}_n$ . The degenerate ground states can be parameterized by the charge  $k = 0, 1, \dots, n-1$  of  $\mathbb{Z}_n$ , being the eigenvectors of

$$(4.18) \quad \tilde{B}_{p_0}^g = \exp\left(\frac{2k\pi gi}{n}\right), T_{p_0}^g = \exp\left(-\frac{2k\pi gi}{n}\right).$$

Such ground states  $\Phi_k$  are related to  $\Phi(\alpha_\partial)$  by the following Fourier transformation

$$(4.19) \quad \Phi_k = \frac{1}{\sqrt{n}} \sum_{\alpha_\partial} \exp\left(\frac{2k\pi \alpha_\partial i}{n}\right) \Phi(\alpha_\partial).$$

One can verify the identity by applying the action of  $T^g$  in Eq. (4.17) directly.

**4.5. On a General Closed Surface.** The de-equivariantization can be applied on an arbitrary closed surface  $Y$  in a similar way. The isomorphism in Eq. (4.10) is replaced by

$$(4.20) \quad \{\alpha_0, \alpha_1, \alpha_2, \dots\}|_{\text{vertex colors}} \cong \{\alpha_0; g_1, g_2, \dots\}|_{\text{trivial homotopy \& trivial holonomy}},$$

where trivial homotopy means that along any non-contractible loop on the dual-triangulation of the graph, the group elements  $g$  multiply to the identity element of  $G$ .

$G$ -symmetric LW models are transformed to traditional LW models in the trivial homotopic Hilbert subspace coupled to a local action. The models are well defined because the Hamiltonian is invariant in the trivial homotopic Hilbert subspace.

## 5. OPEN QUESTIONS

We have studied how Levin-Wen models can be extended to take multi-fusion categories as their input, and how on-site symmetries play a role. There are however still interesting open questions. We mention a few:

- (1) Classify  $n \times n$  **2**-matrices.
- (2) Prove that the LW model with an indecomposable multi-fusion category input  $\mathcal{C} = \oplus_{ij} \mathcal{C}_{ij}$  realizes the Turaev-Viro TQFT based on  $\mathcal{C}_{ii}$  for some  $i$ .
- (3) How to realize symmetry fractionalization, symmetry defects, and gauging with LW models.

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